Proof of the vector triple product equation on page 41.

The triple vector product:

$$\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\, \mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\, \mathbf{w}$$

is described briefly in Chapter 2 and is crucial to some of the theory covered in Chapter 8. The proof of this takes a bit longer than “a few moments of careful algebra” would suggest, so, for completeness, one way of proving it is given below.

Following on from the derivations on p.38 which yielded:

$$\mathbf{v} \wedge \mathbf{w} = (v_y w_z - v_z w_y)\mathbf{i} + (v_z w_x - v_x w_z)\mathbf{j} + (v_x w_y - v_y w_x)\mathbf{k}$$

we can use equations 2.12 to derive the following:

$$\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (v_y w_z - v_z w_y)\, \mathbf{u} \wedge \mathbf{i} + (v_z w_x - v_x w_z)\, \mathbf{u} \wedge \mathbf{j} + (v_x w_y - v_y w_x)\, \mathbf{u} \wedge \mathbf{k}$$

$$= (v_y w_z - v_z w_y)\, (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \wedge \mathbf{i} + (v_z w_x - v_x w_z)\, (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \wedge \mathbf{j} + (v_x w_y - v_y w_x)\, (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \wedge \mathbf{k}$$

$$= (v_y w_z - v_z w_y)\, (-u_y\mathbf{k} + u_z\mathbf{j}) + (v_z w_x - v_x w_z)\, (u_x\mathbf{k} - u_z\mathbf{i}) + (v_x w_y - v_y w_x)\, (-u_x\mathbf{j} + u_y\mathbf{i})$$

$$\begin{aligned} = {} & [- u_z(v_z w_x - v_x w_z) + u_y (v_x w_y - v_y w_x)]\, \mathbf{i} \\ & + [u_z(v_y w_z - v_z w_y) - u_x(v_x w_y - v_y w_x)]\, \mathbf{j} \\ & + [-u_y(v_y w_z - v_z w_y) + u_x(v_z w_x - v_x w_z)]\, \mathbf{k} \end{aligned}$$

Rearranging the terms gives:

$$\begin{aligned} \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = {} & (u_y w_y + u_z w_z)v_x\, \mathbf{i} - (u_y v_y + u_z v_z)w_x\, \mathbf{i} \\ & + (u_x w_x + u_z w_z)v_y\, \mathbf{j} - (u_x v_x + u_z v_z)w_y\, \mathbf{j} \\ & + (u_x w_x + u_y w_y)\, v_z\, \mathbf{k} - (u_x v_x + u_y v_y)w_z\, \mathbf{k} \end{aligned}$$

The final stage of this proof involves the following terms: $u_x v_x w_x\, \mathbf{i} + u_y v_y w_y\, \mathbf{j} + u_z v_z w_z\, \mathbf{k}$ being both added to and subtracted from the preceding equation, thus giving:

$$\begin{aligned} \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = {} & (u_x w_x + u_y w_y + u_z w_z)v_x\, \mathbf{i} - (u_x v_x + u_y v_y + u_z v_z)w_x\, \mathbf{i} \\ & + (u_x w_x + u_y w_y + u_z w_z)v_y\, \mathbf{j} - (u_x v_x + u_y v_y + u_z v_z)w_y\, \mathbf{j} \\ & + (u_x w_x + u_y w_y + u_z w_z)v_z\, \mathbf{k} - (u_x v_x + u_y v_y + u_z v_z)w_z\, \mathbf{k} \end{aligned}$$

We can factor-out the terms in brackets as follows:

$$\begin{aligned} \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) & = (u_x w_x + u_y w_y + u_z w_z)(v_x\, \mathbf{i} + v_y\, \mathbf{j} + v_z\, \mathbf{k}) - (u_x v_x + u_y v_y + u_z v_z)\, (w_x\, \mathbf{i} + w_y\, \mathbf{j} + w_z\, \mathbf{k}) \\ & = (\mathbf{u} \bullet \mathbf{w})\, \mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\, \mathbf{w} \end{aligned}$$