The area under the Dirac delta function

On pages 152 and 154 of “Crystals, X-rays and Proteins” (CXP) by D. Sherwood and J. Cooper (OUP, 2010, 2015) we mention that the area under a Dirac delta function is unity and this is important for its shifting property. The equation for the Dirac delta function is as follows:

$$\delta(x-x')=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk$$

Source: Wikipedia, the free encyclopedia.

When $x = x'$ the exponential term becomes zero and since $e^0 = 1$, the integral becomes an infinitely long sum of 1’s so it has infinite value. When $x \neq x'$ the integral becomes an infinitely long sum of real cosine and imaginary sine terms, both of which oscillate periodically between maxima and minima of plus and minus 1 and they therefore cancel out to zero. Hence the above function has a value of zero everywhere except at $x = x'$ where it has an infinitely high peak. However, it is not immediately obvious why the area under the peak should be unity.

To actually prove this, for reasons that will become clear later on, we need to start by calculating the integral of $\frac{\sin x}{x}$ and this is not easy because integrating it by parts goes on forever. At first, the only article I could find on the web which comes anywhere near to being comprehensible is by G. H. Hardy in the *Mathematical Gazette*, vol. 5 pp. 98-103 (1909) and is entitled “The integral $\int_0^{\infty}\frac{\sin x}{x}dx$.” In Hardy’s article, several different ways of obtaining the integral are given and these are scored by the author for their relative merits! Here we use just the start of the first proof given by Hardy which he describes as “simple and natural.”

I eventually found out that this integral is known as the Dirichlet integral and searching for that phrase then yielded a much richer vein of proofs. Hence, although we use the initial part of Hardy’s first proof, we then divert and follow a shorter and slightly easier one given on Wikipedia (the free encyclopedia) which is, apparently, an example of Feynman’s trick.

$$\boldsymbol{I} \;=\; \int_0^{\infty}\frac{\sin x}{x}dx=\int_0^{\infty}\lim_{a\to 0}\left(e^{-ax}\frac{\sin x}{x}\right)dx=\lim_{a\to 0}\int_0^{\infty}\left(e^{-ax}\frac{\sin x}{x}\right)dx$$

This expansion given by Hardy seems weird because we have introduced the real number a out of the blue and are looking at the integral in the limit as a tends to zero. According to Wikipedia (the free encyclopedia) this is the Laplace transform. Why $-a$ and not just a? Well, maybe it is to stop the result approaching infinity and quite possibly the mathematicians know something we don’t yet know ourselves? The main thing is that as a tends to zero, we approach e^0 which is unity and the dependence on a disappears.

Consider the integral:

$$I=\int_0^{\infty}\left(e^{-ax}\frac{\sin x}{x}\right)dx$$

and remember that to ultimately calculate blue $\boldsymbol{I}$ we actually need to find the value of I as a tends to zero. For now, we perform Feynman’s trick by differentiating I with respect to a as follows:

$$\frac{dI}{da}=\frac{d}{da}\left(\int_0^\infty\left(e^{-ax}\frac{\sin x}{x}\right)dx\right)$$

$$=\int_0^\infty\left(\frac{\partial}{\partial a}(e^{-ax})\frac{\sin x}{x}\right)dx$$

$$=\int_0^\infty\left((-x)(e^{-ax})\frac{\sin x}{x}\right)dx$$

$$=-\int_0^\infty e^{-ax}\sin x\,dx$$

From equation 2.34 of CXP we can substitute the complex exponential form of sin *x*.

$$\frac{dI}{da}=-\int_0^\infty e^{-ax}\left(\frac{e^{ix}-e^{-ix}}{2i}\right)dx$$

$$=-\left(\frac{1}{2i}\right)\int_0^\infty\left(e^{-x(a-i)}-e^{-x(a+i)}\right)dx$$

$$=\frac{1}{2i}\left[\frac{e^{-x(a-i)}}{a-i}-\frac{e^{-x(a+i)}}{a+i}\right]_{x=0}^{x=\infty}$$

Considering the limit $x=\infty$, *e* raised to the power of minus infinity means we are dividing by infinity, so the result is zero. For the other limit when *x* is zero, e^0 is 1. Hence:

$$\frac{dI}{da}=\frac{1}{2i}\left(0-\left(\frac{1}{a-i}-\frac{1}{a+i}\right)\right)$$

$$=-\left(\frac{1}{2i}\right)\left(\frac{1}{a-i}-\frac{1}{a+i}\right)$$

$$=-\left(\frac{1}{2i}\right)\left(\frac{(a+i)-(a-i)}{(a-i)(a+i)}\right)$$

$$=-\left(\frac{1}{2i}\right)\left(\frac{2i}{(a-i)(a+i)}\right)$$

$$=-\left(\frac{1}{a^2-i^2}\right)$$

$$=-\left(\frac{1}{a^2+1}\right)$$

Thus the integral *I* is given by:

$$I = \int \left(\frac{dI}{da} \right) da$$

$$= \int \left(\frac{-1}{a^2 + 1} \right) da$$

On page 21 of "Hints for Advanced Level Mathematics" by L. Harwood Clarke (Heinemann, 1979, 3rd edition), we see from formula 8 that:

$$I = -\arctan(a) + C$$

That is a bit hard to see but we can prove that it is correct by working in reverse. Start by saying that if $y = \arctan(a)$ then $\tan y = a$ and we can then use the chain rule as follows:

$$\frac{d\tan(y)}{da} = \frac{d\tan(y)}{dy} \cdot \frac{dy}{da} = 1$$

$\therefore \ \sec^2 y \dfrac{dy}{da} = 1$ and with a trigonometric expansion $\dfrac{dy}{da} = \dfrac{1}{\sec^2 y} = \dfrac{1}{(1 + \tan^2 y)}$ gives $\dfrac{dy}{da} = \dfrac{1}{(1 + a^2)}$.

Hence the integral of $\dfrac{1}{(a^2+1)}$ is arctan(a) plus an arbitrary constant and the main result so far is:

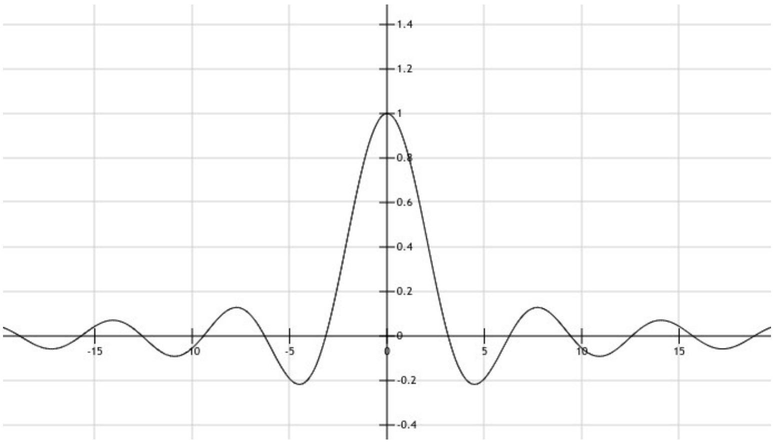
$$I \ = \int \left(\frac{-1}{a^2 + 1} \right) da = -\arctan(a) + C$$

As a tends to infinity, the integral over a in the above equation tends to zero. Hence $I \rightarrow 0$ as $a \rightarrow \infty$. Since the tangent of 90° or $(\pi/2)$ radians is infinity, substituting these values we get the simple equation: $0 = -\dfrac{\pi}{2} + C$ or $C = \dfrac{\pi}{2}$.

$$\therefore \ I \ = \frac{\pi}{2} - \arctan(a)$$

Since $\boldsymbol{I} = \lim\limits_{a \to 0} I$ we can see that $\boldsymbol{I} \ = \ \dfrac{\pi}{2}$ and hence $\displaystyle\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

The following graph shows that $\dfrac{\sin x}{x}$ is a symmetric function.



Hence the area under the curve between x = -∞ and x = 0 must equal that between x = 0 and x = +∞ (which we have just calculated) and the total area under the curve must equal π. Hence:

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx = \pi$$

Since the area under the ripples in the above curve will tend to cancel out as we move away from the origin of the graph, we can see that the total area will be dominated by that of the main central peak. The calculated value for the area of just over 3 therefore seems to agree with the graph.

Going back to the original question of the area under the Dirac delta function, note that the integral is over k rather than x.

$$\delta(x-x') = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} e^{ik(x-x')} dk$$

The following calculation of the integral is adapted from pages 698 – 699 of “Biophysical Chemistry, Part II” by C. R. Cantor and P. R. Schimmel (Freeman, 1980). Since the delta function has a value of zero everywhere except when $x = x'$, we can calculate the area under it over an arbitrary range of x values which includes the peak, say from $x' - 1$ to $x' + 1$ as below:

$$\int_{x'-1}^{x'+1} \delta(x-x') dx = \left(\frac{1}{2\pi}\right) \int_{x'-1}^{x'+1} \left(\int_{-\infty}^{\infty} e^{ik(x-x')} dk \right) dx$$

We can then swap the order of the integration because the variables x and k are independent e.g. see Section 4.4 of “Mathematics for Chemists” by P. G. Francis (Chapman and Hall, 1984) which is on multiple integration. Multiplying through by 2π as well gives:

$$2\pi \int_{x'-1}^{x'+1} \delta(x-x') dx = \int_{-\infty}^{\infty} \left(\int_{x'-1}^{x'+1} e^{ik(x-x')} dx \right) dk$$

$$= \int_{-\infty}^{\infty} e^{-ikx'} \left(\int_{x'-1}^{x'+1} e^{ikx} dx \right) dk$$

$$= \int_{-\infty}^{\infty} e^{-ikx'} \left(\frac{1}{ik}\right) \left(e^{ik(x'+1)} - e^{ik(x'-1)}\right) dk$$

$$= \int_{-\infty}^{\infty} e^{-ikx'} e^{ikx'} \left(\frac{1}{ik}\right) \left(e^{ik} - e^{-ik}\right) dk$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{ik}\right) \left(e^{ik} - e^{-ik}\right) dk$$

Equation 2.34 of CXP gives us:

$$2\pi \int_{x'-1}^{x'+1} \delta(x-x') dx \; = \; \int_{-\infty}^{\infty} \left(\frac{2i}{ik}\right) \sin(k) dk \; = \; 2 \int_{-\infty}^{\infty} \frac{\sin(k)}{k} dk$$

Hence:

$$\int_{x'-1}^{x'+1} \delta(x-x')\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(k)}{k}\,dk$$

Since the delta function is zero everywhere except at $x = x'$ we can expand the range of the integral on x as follows:

$$\int_{-\infty}^{+\infty} \delta(x-x')\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(k)}{k}\,dk$$

Based on the previous proof we can see that:

$$\int_{-\infty}^{+\infty} \frac{\sin k}{k}\,dk = \pi$$

Hence:

$$\int_{-\infty}^{+\infty} \delta(x-x')\,dx = 1$$
