The area under the Dirac delta function

On pages 152 and 154 we mention that the area under a Dirac delta function is unity and this is important for its shifting property. The equation for the Dirac delta function is as follows:

$$\delta(x-x')=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk$$

Source: Wikipedia, the free encyclopedia.

When $x = x'$ the exponential term becomes zero and since $e^0 = 1$, the integral becomes an infinitely long sum of 1's so it has infinite value. When $x \neq x'$ the integral becomes an infinitely long sum of real cosine and imaginary sine terms, both of which oscillate periodically between minima and maxima of plus and minus 1 and they therefore cancel out to zero. Hence the above function has a value of zero everywhere except at $x = x'$ where it has an infinitely high peak. However, it is not immediately obvious why the area under the peak should be unity.

To actually prove this, for reasons that will become clear later on, we need to start by calculating the integral of $\frac{\sin x}{x}$ and this is not easy because integrating it by parts goes on forever. The only article I could find on the web which comes anywhere near to being comprehensible is by G. H. Hardy in the *Mathematical Gazette*, vol. 5 pp. 98-103 (1909) and is entitled "The integral $\int_0^{\infty} \frac{\sin x}{x}\,dx$." In Hardy's article, several different ways of obtaining the integral are given and these are scored by the author for their relative merits! The one that is reproduced here is the first one given by Hardy which he describes as "simple and natural." It achieves a mid-range score.

$$\boldsymbol{I} \;=\; \int_0^{\infty} \frac{\sin x}{x}\,dx = \int_0^{\infty} \lim_{a\to 0}\left(e^{-ax}\frac{\sin x}{x}\right)dx = \lim_{a\to 0}\int_0^{\infty}\left(e^{-ax}\frac{\sin x}{x}\right)dx$$

This expansion by Hardy seems weird because we have introduced the real number *a* out of the blue and are looking at the integral in the limit as *a* tends to zero. Why -*a* and not just *a*? Well, maybe it is to stop the result approaching infinity and quite possibly the author knows something we don't yet know ourselves? The main thing is that as *a* tends to zero, we approach e^0 which is unity and the dependence on *a* disappears. Anyway we then introduce another variable, *t*.

$$\boldsymbol{I} \;=\; \lim_{a\to 0}\int_0^{\infty} e^{-ax}\,dx \int_0^{1} \cos tx\,dt$$

In the term $\int_0^1 \cos tx\,dt$ we treat *x* as a constant and by page 21 of L. Harwood Clarke's "Hints for Advanced Level Mathematics" (Heinemann, 1979) we see that this is indeed equal to $\left[\frac{\sin tx}{x}\right]_{t=0}^{t=1}$ or $\frac{\sin x}{x}$ which is quite neat. Swapping around the order of the integration is fine because the variables are independent e.g. see Section 4.4 of "Mathematics for Chemists" by P. G. Francis (Chapman and Hall, 1984) which is on multiple integration.

$$\boldsymbol{I} \;=\; \lim_{a\to 0}\int_0^{1} dt \int_0^{\infty} e^{-ax}\cos tx\,dx$$

Let's try doing just the right-hand integral over x, *i.e.* $I=\int e^{-ax}\cos tx\,dx$ by parts using the general formula:

$$I = \int u\,v\,dx = u\int v\,dx - \int \left(\int v\,dx \cdot \frac{du}{dx}\right) dx$$

with $u=e^{-ax}$ and $v=\cos tx$. Hence:

$$I = e^{-ax}\int \cos tx\,dx - \int \left(\int \cos tx\,dx \cdot (-a)\,e^{-ax}\right) dx$$

$$= e^{-ax}\frac{\sin tx}{t} - \int \left(\frac{\sin tx}{t}\cdot(-a)\,e^{-ax}\right) dx$$

$$= \frac{1}{t}e^{-ax}\sin tx + \frac{a}{t}\int e^{-ax}\sin tx\,dx$$

Repeat for second term with $u=e^{-ax}$ and $v=\sin tx$.

$$I = \frac{1}{t}e^{-ax}\sin tx + \frac{a}{t}\left(e^{-ax}\int \sin tx\,dx - \int \left(\int \sin tx\,dx\cdot(-a)e^{-ax}\right) dx\right)$$

$$= \frac{1}{t}e^{-ax}\sin tx + \frac{a}{t}\left(e^{-ax}\left(\frac{-\cos tx}{t}\right) - \int \left(\frac{-\cos tx}{t}\right)(-a)\,e^{-ax}\,dx\right)$$

$$= \frac{1}{t}e^{-ax}\sin tx + \frac{a}{t^2}\left(-e^{-ax}\cos tx - a\int e^{-ax}\cos tx\,dx\right)$$

$$= \frac{1}{t}e^{-ax}\sin tx - \frac{a}{t^2}\left(e^{-ax}\cos tx + a\int e^{-ax}\cos tx\,dx\right)$$

$$= \frac{1}{t}e^{-ax}\sin tx - \frac{a}{t^2}\left(e^{-ax}\cos tx + a\,I\right)$$

$$I = \frac{1}{t}e^{-ax}\sin tx - \frac{a}{t^2}e^{-ax}\cos tx - \frac{a^2}{t^2}I$$

$$\therefore\ I + \frac{a^2}{t^2}I = \frac{1}{t}e^{-ax}\sin tx - \frac{a}{t^2}e^{-ax}\cos tx$$

$$I\left(\frac{t^2+a^2}{t^2}\right) = \frac{1}{t}e^{-ax}\sin tx - \frac{a}{t^2}e^{-ax}\cos tx$$

$$\therefore\ I = \left(\frac{t}{t^2+a^2}\right)e^{-ax}\sin tx - \left(\frac{a}{t^2+a^2}\right)e^{-ax}\cos tx$$

$$I = \left(\frac{e^{-ax}}{t^2+a^2}\right)(t\sin tx - a\cos tx)$$

Octave symbolic gives the same result!

```
                                   -a·x
(-a·cos(t·x) + t·sin(t·x))·e
─────────────────────────────────
               2    2
              a  + t
```

Note that strictly speaking there should be an arbitrary constant on the right-hand side of the above expression for *I* which we have omitted for convenience. However this constant disappears when we substitute the limits for *x* as below.

$$[I]_{x=0}^{x=\infty}=\left[\left(\frac{e^{-ax}}{t^2+a^2}\right)(t\sin tx\,-\,a\cos tx)\right]_{x=0}^{x=\infty}$$

and rearranging gives:

$$[I]_{x=0}^{x=\infty}=\left[\frac{1}{e^{ax}(t^2+a^2)}(t\sin tx\,-\,a\cos tx)\right]_{x=0}^{x=\infty}$$

Considering the limit $x=\infty$, the sin or cos of infinity will be somewhere between plus and minus 1 but *e* raised to the power of infinity in the denominator means we are dividing by infinity, so the result is zero. For the other limit when *x* is zero, e^0 is 1 and the sin term disappears while the cos term is unity. Hence:

$$[I]_{x=0}^{x=\infty}\,=\,0-\left(\frac{-a}{t^2+a^2}\right)\,=\,\frac{a}{t^2+a^2}$$

Octave gives the same result with a couple of clever conditions which do not matter if *a* and *t* are real numbers, which I was assuming to be the case anyway!

```
   a                                        π
───────      for │arg(t)│ = 0 ∧ │arg(a)│ < ─
 2    2                                     2
a  + t
```

Returning to our formula for the integral of $\frac{\sin x}{x}$ or blue ***I*** namely:

$$I\;=\;\lim_{a\to 0}\int_0^1 dt\int_0^\infty e^{-ax}\cos tx\,dx$$

we have been able to simplify this to:

$$I\;=\;\lim_{a\to 0}\int_0^1 \frac{a}{t^2+a^2}\,dt$$

On page 21 of L. Harwood Clarke, we see from formula 16 that:

$$\boldsymbol{I} = \lim_{a \to 0} \int_0^1 \frac{a}{t^2 + a^2} \, dt = \lim_{a \to 0} \left[\arctan\left(\frac{t}{a}\right) \right]_{t=0}^{t=1}$$

That is a bit hard to see but we can prove that it is correct by working in reverse. Start by saying that if $y = \arctan(t/a)$ then $\tan y = (t/a)$ and we can then use the chain rule as follows:

$$\frac{d\tan(y)}{dt} = \frac{d\tan(y)}{dy} \cdot \frac{dy}{dt} = \frac{1}{a}$$

$\therefore \sec^2 y \frac{dy}{dt} = \frac{1}{a}$ and with a trigonometric expansion $\frac{dy}{dt} = \frac{1}{a\sec^2 y} = \frac{1}{a(1+\tan^2 y)}$ we then get:

$$\frac{dy}{dt} = \frac{1}{a(1+(t/a)^2)} = \frac{a}{(a^2+t^2)}$$

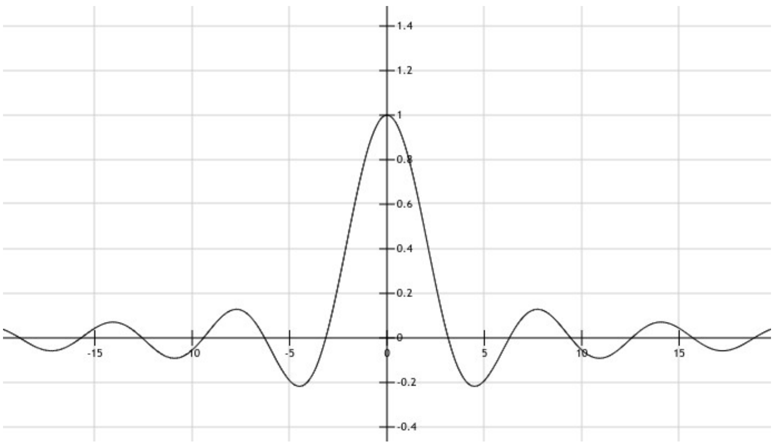
Hence we have proved that integrating $\frac{a}{(a^2+t^2)}$ over t gives $\arctan\left(\frac{t}{a}\right)$ plus an arbitrary constant which disappears when we apply the limits of integration as below.

$$\boldsymbol{I} = \lim_{a \to 0} \left[\arctan\left(\frac{t}{a}\right) \right]_{t=0}^{t=1} = \lim_{a \to 0} \arctan\left(\frac{1}{a}\right)$$

As $a \to 0$, $(1/a)$ tends to infinity which is the tangent of 90 ° or $(\pi/2)$ radians.

$$\therefore \boldsymbol{I} = \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

The following graph shows that $\frac{\sin x}{x}$ is a symmetric function.



Hence the area under the curve between x = -∞ and x = 0 must equal that between x = 0 and x = +∞ (which we have just calculated) and the total area under the curve must equal π. Hence:

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx = \pi$$

Going back to the original question of the area under the Dirac delta function, note that the integral is over *k* rather than *x*.

$$\delta(x-x')=\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} e^{ik(x-x')}dk$$

The following calculation of the integral is adapted from pages 698 – 699 of "Biophysical Chemistry, Part II" by C. R. Cantor and P. R. Schimmel (Freeman, 1980). Since the delta function has a value of zero everywhere except when $x = x'$, we can calculate the area under it over an arbitrary range of *x* values which includes the peak, say from $x' - 1$ to $x' + 1$ as below:

$$\int_{x'-1}^{x'+1} \delta(x-x')\,dx=\left(\frac{1}{2\pi}\right)\int_{x'-1}^{x'+1}\left(\int_{-\infty}^{\infty} e^{ik(x-x')}dk\right)dx$$

We can multiply through by 2π and swap the order of integration giving:

$$2\pi\int_{x'-1}^{x'+1} \delta(x-x')\,dx=\int_{-\infty}^{\infty}\left(\int_{x'-1}^{x'+1} e^{ik(x-x')}dx\right)dk$$

$$=\int_{-\infty}^{\infty} e^{-ikx'}\left(\int_{x'-1}^{x'+1} e^{ikx}dx\right)dk$$

$$=\int_{-\infty}^{\infty} e^{-ikx'}\left(\frac{1}{ik}\right)\left(e^{ik(x'+1)}-e^{ik(x'-1)}\right)dk$$

$$=\int_{-\infty}^{\infty} e^{-ikx'}e^{ikx'}\left(\frac{1}{ik}\right)\left(e^{ik}-e^{-ik}\right)dk$$

$$=\int_{-\infty}^{\infty}\left(\frac{1}{ik}\right)\left(e^{ik}-e^{-ik}\right)dk$$

Equation 2.34 in Chapter 2 gives us

$$2\pi\int_{x'-1}^{x'+1} \delta(x-x')\,dx \;=\; \int_{-\infty}^{\infty}\left(\frac{2i}{ik}\right)\sin(k)\,dk \;=\; 2\int_{-\infty}^{\infty}\frac{\sin(k)}{k}dk$$

Hence:

$$\int_{x'-1}^{x'+1} \delta(x-x')\,dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(k)}{k}dk$$

Since the delta function is zero everywhere except at $x = x'$ we can expand the range of the integral on *x* as follows:

$$\int_{-\infty}^{+\infty} \delta(x-x')\,dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(k)}{k}dk$$

Based on the previous proof we can see that:

$$\int_{-\infty}^{+\infty} \frac{\sin k}{k} dk = \pi$$

Hence:

$$\int_{-\infty}^{+\infty} \delta(x - x') dx = 1$$
